Higher Poisson manifolds and higher Koszul–Schouten brackets

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Introduction

Higher Poisson manifolds?

- Classical Poisson structures are bi-vectors.
- Higher Poisson structures are more general (inhomogenous) multivector fields.

Aim of talk

Describe

- Higher Poisson brackets on functions.
- Higher Koszul–Schouten brackets on differential forms.

Differential forms, multivectors and Cartan calculus

Antitangent bundle, ΠTM

$\{x^A, dx^A\}$.

$$x^A \rightarrow \overline{x}^A(x), \qquad dx^A \rightarrow d\overline{x}^A = dx^B \left(\frac{\partial \overline{x}^A}{\partial x^B}\right).$$

Anticotangent bundle, ΠT^*M

$\{x^A, x^*_A\}$.

$$x^A \rightarrow \overline{x}^A(x), \qquad x^*_A \rightarrow \overline{x}^*_A = \left(\frac{\partial x^B}{\partial \overline{x}^A}\right) x^*_B.$$

Define

$\Omega^*(M) = C^\infty(\Pi TM)$ and $\mathfrak{X}^*(M) = C^\infty(\Pi T^*M)$

Differential forms, multivectors and Cartan calculus

The **Cartan calculus**;

1. Exterior derivative $d = dx^A \frac{\partial}{\partial x^A}$
2. Interior product $i_X = (-1)^{\widetilde{X}+1} X(x, \partial_{dx})$
3. Lie derivative $L_X = [d, i_X]$

with $X = X(x, x^*) \in \mathfrak{X}^*(M)$.

The **Schouten–Nijenhuis bracket**;

$$[\![X, Y]\!] = (-1)^{(\widetilde{A}+1)(\widetilde{X}+1)} \frac{\partial X}{\partial x^*_A} \frac{\partial Y}{\partial x^A} - (-1)^{\widetilde{A}(\widetilde{X}+1)} \frac{\partial X}{\partial x^A} \frac{\partial Y}{\partial x^*_A},$$

Important result;

$[L_X, L_Y] = L_{[\![X,Y]\!]}$.

$L_\infty - algebras$

A series of odd brackets on vector space s.t.

1. The operators are symmetric

$$(a_1, a_2, \cdots, a_i, a_j, \cdots, a_n) = (-1)^{\widetilde{a}_i \widetilde{a}_j}(a_1, a_2, \cdots, a_j, a_i, \cdots, a_n).$$

2. The generalised Jacobi identites or Jacobiator

$$\sum_{k+l=n} \sum_{(k,l)-shuffels} \pm \left((a_{\sigma(1)}, \cdots, a_{\sigma(k)}), a_{\sigma(k+1)}, \cdots, a_{\sigma(k+1)}\right) = 0,$$

hold for all n.

A (k, l)-shuffle is a permutation of the indices $1, 2, \cdots k + l$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

L_∞ – algebras

The first few Jacobiators are;

$$
\begin{aligned}
J^0 &= ((\emptyset)) = 0\\
J^1 &= ((a)) + ((\emptyset), a) = 0\\
J^2 &= ((a,b)) + ((a), b) + (-1)^{\widetilde{a}\widetilde{b}}((b), a) + ((\emptyset), a, b) = 0\\
J^3 &= ((a,b,c)) + ((a,b),c) + (-1)^{\widetilde{b}\widetilde{c}}((a,c),b)\\
&+ (-1)^{\widetilde{a}(\widetilde{b}+\widetilde{c})}((b,c),a) + ((a),b,c) + (-1)^{\widetilde{a}\widetilde{b}}((b),a,c)\\
&+ (-1)^{(\widetilde{a}+\widetilde{b})\widetilde{c}}((c),a,b) + ((\emptyset),a,b,c) = 0
\end{aligned}
$$

Warning other conventions on grading and symmetry exist.

Higher Poisson structures and brackets

Definition

A manifold M equipped with an even multivector field $P \in \mathfrak{X}^*(M)$ such that $[\![P, P]\!] = 0$, is as a **higher Poisson manifold**.

Definition

The **higher Poisson brackets** defined by

$\{f_1, f_2, \cdots, f_r\}_P = [\![\cdots [\![[\![P, f_1]\!], f_2]\!], \cdots, f_r]\!]|_M$,
with $f_I \in C^\infty(M)$.

The higher Poisson brackets form an L_∞-algebra.

Jacobiators $= 0 \quad \Leftrightarrow \quad [\![P, P]\!] = 0$.

Higher Koszul–Schouten brackets

Definition

The **higher Koszul–Schouten brackets** are defined by

$$[\alpha_1, \alpha_2, \cdots, \alpha_r]_P = [\cdots[[L_P, \alpha_1], \alpha_2], \cdots, \alpha_r]\mathbb{1},$$

where $\alpha_I \in \Omega^*(M)$ and $\mathbb{1}$ is the identity zero form.

The higher Koszul–Schouten brackets form an L_∞-algebra.

$$\text{Jacobiators} = 0 \quad \Leftrightarrow \quad (L_P)^2 = \frac{1}{2}[L_P, L_P] = \frac{1}{2}L_{[\![P,P]\!]} = 0.$$

Relation between the higher Possion and Koszul–Schouten brackets

Theorem

The higher Koszul–Schouten brackets and the higher Poisson brackets satisfy the following;

1. $[f_1, df_2 \cdots, df_r]_P = \{f_1, f_2, \cdots, f_r\}_P,$
2. $[f_1, f_2, \cdots, f_r]_P = 0 \qquad \text{for } r > 1,$
3. $[\emptyset]_P = -d\{\emptyset\}_P,$
4. $[df_1, df_2, \cdots, df_r]_P = -d\{f_1, f_2, \cdots, f_r\}_P,$

Warning There is a proviso that P is polynomial in x^*

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Relation to BV-antifield formulism

Finite dimensional “model"

antifields $\leftrightarrow$ anticoordinate
extended classical action $\leftrightarrow$ higher Poisson structure .
antibracket $\leftrightarrow$ Schouten–Nienhuis bracket.

Then we have an L_∞-algebra structure on;

- the space of field and ghost valued functions (goes back to Jim Stasheff),
- the space of field and ghost valued differential forms (appears new).

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