Brane Tilings, M2-Branes and Chern-Simons Theories

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My Collaborators

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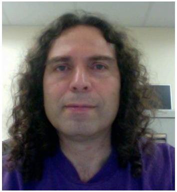

- Special thanks to: Yang-Hui He, Alexander Shannon, and Ed Segal



Part I: Motivation and Introduction

What is an M2-brane?

- Example from EM: A charged particle moving along a 1 dimensional worldline is a source of 1-form field A_μ.
- In supergravity, a p-brane is a $(p+1)$ space-time dimensional object sourcing the $(p+1)$-form gauge field.
- In 11d SUGRA, the only antisymmetric tensor field is the 3-form $A^{(3)}$. The corresponding field strength is a 4-form $F^{(4)} = dA^{(3)}$.
  - Maxwell eq. for an electric source: $\underbrace{d\overbrace{*F^{(4)}}^{7-\text{form}}}_{8-\text{form}} = *\delta^{(3)}$

    $\Rightarrow$ Elec. charge is localised in 3 $(= 2+1)$ spacetime dim. $\Rightarrow$ M2-brane.
  - Maxwell eq. for a magnetic source: $\underbrace{dF^{(4)}}_{5-\text{form}} = *\delta^{(6)}$

    $\Rightarrow$ Mag. charge is localised in 6 $(= 5+1)$ spacetime dim. $\Rightarrow$ M5-brane.

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Motivation

- How many conformal field theories (CFTs) do we know in $(2+1)$ dimensions?
- What are the worldvolume theories of a stack of N M2-branes in M-theory?
- Understand Chern-Simons (CS) theories better
- Algebraic Geometry & Quiver Gauge Theories

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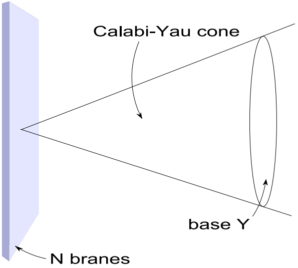
Motivation: AdS/CFT

Long standing problem:

- What is the field theory dual to the M-theory in $\mathrm{AdS}_4 \times \mathrm{Y}_7$ background? (Y_7 is a Sasaki–Einstein 7-manifold)
- Each Y_7 leads to a different CFT
- The field theory can be realised as a worldvolume theory of N M2-branes placed at the tip of the Calabi–Yau cone over Y_7



Part II: $\mathcal{N} = 2$ CS-Matter Theories

"Theories with $\mathcal{N} = 1$ supersymmetry in three dimensions have no holomorphy properties, so we cannot control their non-perturbative dynamics."

[Aharony, Hanany, Intriligator, Seiberg, Strassler '97]

$\mathcal{N} = 2$ CS-Matter Theories

- Gauge group: $\mathcal{G} = \prod_{a=1}^{G} U(N_a)$
- A 3d $\mathcal{N} = 2$ vector multiplet V_a can be obtained from a dimensional reduction of 4d $\mathcal{N} = 1$ vector multiplet. It consists of
  - A one-form gauge field A_a , a real scalar field σ_a (from the components of the vector field in the compactified direction) , a two-component Dirac spinor χ_a , a real auxiliary scalar fields D_a.
  - All fields transform in the adjoint representation of $U(N_a)$.
- Matter fields are denoted by Φ_{ab}. Each of them is a chiral multiplet accordingly charged in the gauge groups $U(N_a)$ and $U(N_b)$. It consists of
  - Complex scalars X_{ab} , Fermions ψ_{ab} , Auxiliary scalars F_{ab} .

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$\mathcal{N} = 2$ CS-Matter Lagrangian

- The action consists of 3 terms: $S = S_{\rm CS} + S_{\rm matter} + S_{\rm potential}$.
- CS term in Wess–Zumino gauge:

$$S_{\rm CS} = \sum_{a=1}^{G} \frac{k_a}{4\pi} \int {\rm Tr} \left(A_a \wedge {\rm d}A_a + \frac{2}{3} A_a \wedge A_a \wedge A_a - \bar{\chi}_a \chi_a + 2 D_a \sigma_a \right) ,$$

  where k_a are called the **CS levels**.
- The matter (kinetic) term is

$$S_{\rm matter} = \int {\rm d}^3x \ {\rm d}^4\theta \sum_{\Phi_{ab}} {\rm Tr} \left(\Phi^\dagger_{ab} e^{-V_a} \Phi_{ab} e^{V_b} \right) .$$

- The superpotential term is

$$S_{\rm potential} = \int {\rm d}^3x \ {\rm d}^2\theta \, W(\Phi_{ab}) + {\rm c.c.} \ .$$

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What Is Special in $2 + 1$ dimensions?

- The Yang–Mills coupling has mass dimension $1/2$ in $(2 + 1)$ dimensions
  - All theories are strongly coupled in the IR
- The CS levels k_a are integer valued (to ensure gauge invariance of the action)
  - **Non-renormalisable theorem (NRT):** Each k_a is not renormalised beyond a possible finite 1-loop shift [Witten '99]
- The CS levels k_a have mass dimensions 0
  - All couplings in the action are classically marginal
  - NRT $\Rightarrow$ All couplings are quantum mechanically exactly marginal

    (Quantum corrections are either irrelevant in the IR or can be absorbed by field redefinitions.)
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The Mesonic Moduli Space

- The vacuum equations:
  - F-terms: $\partial_{X_{ab}} W = 0$
  - 1st D-terms: $\mu_a(X) := \sum_{b=1}^{G} X_{ab} X_{ab}^\dagger - \sum_{c=1}^{G} X_{ca}^\dagger X_{ca} + [X_{aa}, X_{aa}^\dagger] = 4k_a \sigma_a$
  - 2nd D-terms: $\sigma_a X_{ab} - X_{ab} \sigma_b = 0$.
  - Note that the fields X_{ab}, σ_a are matrices, and no summation convention.
- Space of solutions of these eqns are called the mesonic moduli space, $\mathcal{M}^{\text{mes}}$.

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Relation to M2-branes

- Assume that
  1. Gauge group: $\mathcal{G} = U(N)^G$ (*i.e.* setting all $N_a = N$)
  2. Each chiral multiplet appears precisely twice in W. Once with a positive sign and once with a negative sign. (toric condition)
- Consequences:
  1. N has the physical interpretation as the number of M2-branes in the stack on which the gauge theory is living
  2. The mesonic moduli space $\mathcal{M}^{\rm mes}$ is in fact the space that an M2-brane probes
  3. The mesonic moduli space is 4 complex dimensional. It is a CY 4-fold.

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Quiver Gauge Theories

What is a quiver gauge theory?

- It is a gauge theory which can be represented by a directed graph with nodes and arrows.
  - Each node represents each factor in the gauge group $\mathcal{G}$.
  - Each arrow going from a node a to a different node b represents a field X_{ab} in the bifundamental rep. $(\mathbf{N}, \overline{\mathbf{N}})$ of $U(N)_a \times U(N)_b$.
  - Each loop on a node a represents a field ϕ_a in the adjoint rep. of $U(N)_a$.

- For a CS quiver theory, we also need to assign the CS levels k_a to each node.

Abelian CS Quiver Theories

- Take $N = 1$. Gauge group $\mathcal{G} = U(1)^G$.
- The fields X_{ab}, σ_a are just **complex numbers**.
- The **vacuum equations** do the following things:
  - Set all σ_a to a single field, say σ. It is a real field.
  - Impose the following condition on the CS levels: $\sum_a k_a = 0$.
- For simplicity, take $k \equiv \gcd(\{k_a\}) = 1$. Otherwise, simply consider the $\mathbb{Z}_k$ orbifold of the mesonic moduli space.

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Moduli Space of a CS Quiver Theory

Let's consider first the abelian case $N = 1$.

- Solving the vacuum equations in 2 steps:
  1. Solving F-terms. The space of solutions of F-terms is the Master space, $\mathcal{F}^\flat$.
  2. Further solving D-terms: Modding out $\mathcal{F}^\flat$ by **the gauge symmetry**.
- Among the original gauge symmetry $U(1)^G$, one is a diagonal $U(1)$; it does not couple to matter fields $\rightarrow$ We are left with $U(1)^{G-1}$.
- Up to this point, the process is the same for a (3+1)d theory living on a D3-brane probing CY_3

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Moduli Space of a CS Quiver Theory

- The CS levels induce FI-like terms: $\zeta_a = 4k_a\sigma$; it selects out another $U(1)$ to fibre over CY_3 to give a CY_4.
- The mesonic moduli space $\mathcal{M}^{\mathrm{mes}}$ is a CY_4.

  $\rightarrow$ We are left with $U(1)^{G-2}$. This gives $G-2$ baryonic directions.
- Therefore, the mesonic moduli space can be written as

$$\mathcal{M}^{\mathrm{mes}}_{N=1} = \mathcal{F}^{\flat}//U(1)^{G-2}$$

- For higher N, we simply have

$$\mathcal{M}^{\mathrm{mes}}_{N} = \mathrm{Sym}^N\left(\mathcal{M}^{\mathrm{mes}}_{N=1}\right)$$

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- Therefore, the mesonic moduli space can be written as

$$\mathcal{M}^{\mathrm{mes}}_{N=1} = \mathcal{F}^{\flat}//U(1)^{G-2}$$

- For higher N, we simply have

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Moduli Space of a CS Quiver Theory

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Example: The ABJM Theory

- The theory has $G = 2$ gauge groups, and 4 bi-fundamental fields $X_{12}^1, X_{12}^2, X_{21}^1, X_{21}^2$. [Aharony, Bergman, Jafferis, Maldacena '08]

- The CS levels: $k_1 = 1, \; k_2 = -1$.
- Superpotential: $W = \mathrm{Tr}(X_{12}^1 X_{21}^1 X_{12}^2 X_{21}^2 - X_{12}^1 X_{21}^2 X_{12}^2 X_{21}^1)$.
- The abelian case ($N = 1$): $W = 0$

$\Rightarrow$ The F-terms admit any complex solutions of X_{12}^i, X_{21}^i $(i = 1, 2)$

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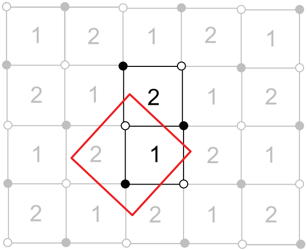
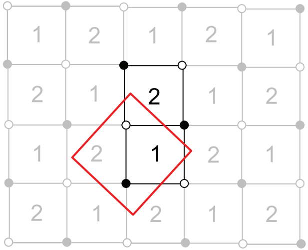
Part III: Brane Tilings

Brane Tilings

- The toric condition of the superpotential gives rise to a bipartite graph on $\mathbb{T}^2$ which is also known as a brane tiling. (Hanany *et al.*)
- For a $(2+1)$-dimensional theory, the tiling has an interpretation of a network of D4-branes and NS5-brane ending on the NS5-brane in Type IIA (which is a compactification of M-theory). (Imamura & Kimura '08)
- **Example:** The quiver diagram and the brane tiling of the ABJM Theory

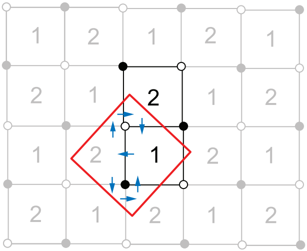


Tiling-Quiver Dictionary



- $2n$ **sided face** $=$ $U(N)$ gauge group with nN flavours
- **Edge** $=$ A chiral field charged under the two gauge group corresponding to the faces it separates
- D **valent node** $=$ A D-th order interaction term in superpotential

Comments on Brane Tilings



- **Graph is bipartite:** Nodes alternate between clockwise (white) and anticlockwise (black) orientations of arrows.
- Black (white) nodes connected to white (black) only
- Odd sided faces are forbidden by anomaly cancellation condition
- White (black) nodes give $+$ $(-)$ sign in the superpotential

  Example of ABJM: $W = \mathrm{Tr}(X^1_{12}X^1_{21}X^2_{12}X^2_{21} - X^1_{12}X^2_{21}X^2_{12}X^1_{21})$

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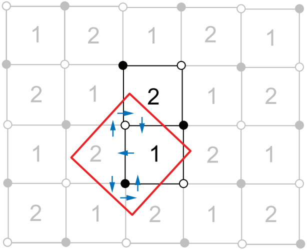

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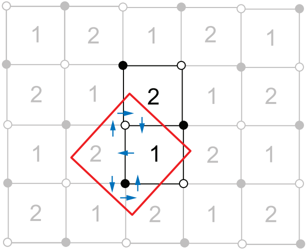

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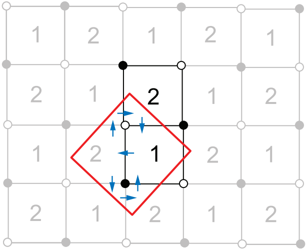

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Part IV: Toric Phases

Current Results for $(2+1)$d Theories

- Each brane tiling (with specified CS levels) defines a unique Lagrangian for an $\mathcal{N} = 2$ CS theory in 2+1 dimensions.
- All models described by brane tilings are conjectured to live on the worldvolume of an M2-brane probing the CY4
- Largest known family of SCFTs in $(2+1)$ dimensions!
- Tiling information (= quiver + superpotential + CS levels) $\longrightarrow \mathcal{M}^{\text{mes}}$ via the **'forward algorithm'** [Hanany *et al.*]

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Toric Phases

- There are some models which have different brane tilings, but have the same mesonic moduli space. [Davey, Hanany, He, NM, Torri '08 - '09]
- These models are said to be **toric dual** to each other. Each of these models is referred to as **toric phase**.
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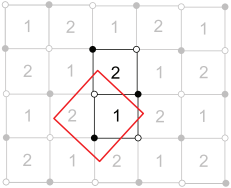
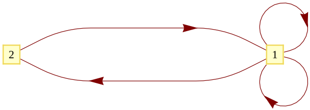
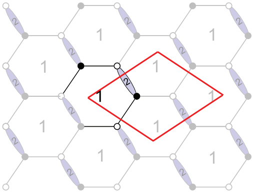
Phases of The $\mathbb{C}^4$ Theory

- Phase I: The ABJM model $(k_1 = -k_2 = 1)$



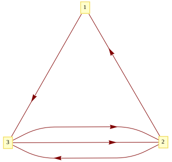
- Phase II: The dual ABJM model $(k_1 = -k_2 = 1)$

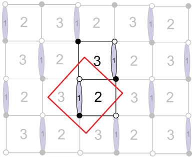




Phases of The Conifold $(\mathcal{C}) \times \mathbb{C}$ Theory

- Phase I: $k_1 = -k_2 = 1, k_3 = 0$





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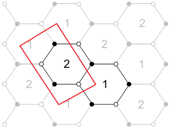
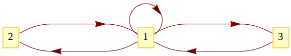
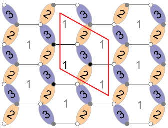


- Phase III: $k_1 = 0, k_2 = -k_3 = 1$





Phases of The D_3 Theory

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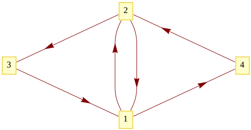

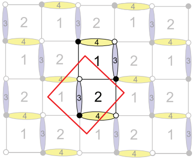

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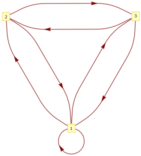

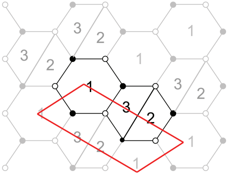

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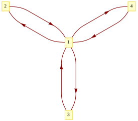

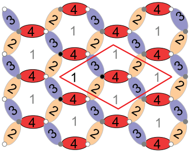

Phases of The $Q^{1,1,1}/\mathbb{Z}_2$ Theory

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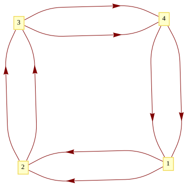

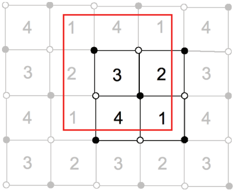

- Phase II: $k_1 = k_2 = -k_3 = -k_{3'} = 1$